Bandits and Reinforcement Learning

COMS E6998.001 Fall 2017

Columbia University

Alekh Agarwal and Alex Slivkins
Microsoft Research NYC

Probability and concentration recap

Many of the slides adopted from Ron Jin and Mohammad Hajiaghayi

Outline

- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

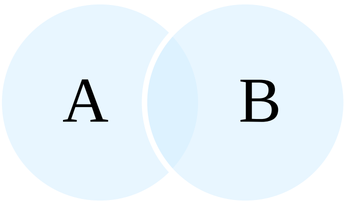
Random events

- ***Experiment***: e.g.: toss a coin twice
- ***Sample space***: possible outcomes of an experiment
  - S = {HH, HT, TH, TT}
- ***Event***: a subset of possible outcomes
  - A={HH}, B={HT, TH}
  - complement $\bar{A}$={HT, TH, TT}
  - disjoint (mutually exclusive) events: if $A \cap B = \emptyset$.
- Shorthand:
  - AB for $A \cap B$
- For now: ***assume finite #outcomes***

Definition of Probability

- ***Probability of an outcome u:***
  a number assigned to u**,** $\Pr(u) \geq 0$
  - Two coin tosses: {HH, HT, TH, TT}
    each outcome has probability ¼.
  - Axiom: $\sum_{u \in S} \Pr(u) \ = 1$
- ***Probability of an event*** $\boldsymbol{A} \subset \boldsymbol{S}$:
  a number assigned to event: $\Pr(A) = \sum_{u \in A} \Pr(u)$
- ***Probability space***:
  - sample space S
  - probability $\Pr(u)$ for each outcome $u \in S$

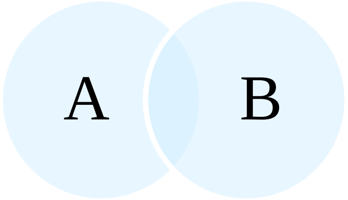
Joint Probability



- For events A and B,
  joint probability Pr(AB) (also written as Pr(A ∩ B))
  is the probability that both events happen.
- Example: A={HH}, B={HT, TH},
  what is the joint probability Pr(AB)?

  Zero

Independence



- Two events ***A and B are independent*** if

  $$\Pr(AB) = \Pr(A)\Pr(B)$$

  "Occurrence of *A* does not affect the probability of *B*"

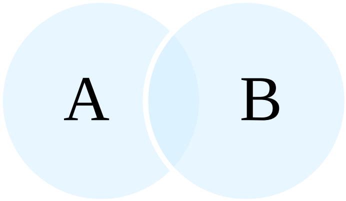
  - **Prop:** $\Pr(\bar{\boldsymbol{A}}B)=\Pr(\bar{\boldsymbol{A}})\Pr(B)$
  - Proof: $\Pr(AB)+\Pr(\bar{\boldsymbol{A}}B)=\Pr(B)$

    $$\begin{aligned}\Pr(\bar{\boldsymbol{A}}B) &= \Pr(B)-\Pr(AB)\\ &= \Pr(B)-\Pr(A)\Pr(B)\\ &= \Pr(B)(1-\Pr(A)) = \Pr(B)\Pr(\bar{\boldsymbol{A}}).\end{aligned}$$

- Events $\{A_i\}$ are ***mutually independent*** in case

  $$\Pr\left(\bigcap_i A_i\right) = \prod_i \Pr(A_i)$$

Independence: examples

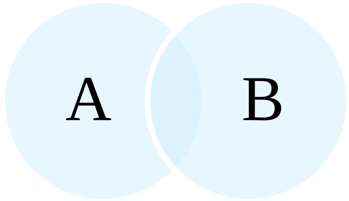


- Recall A and B are independent if Pr(AB) = Pr(A)Pr(B)
- Example: Medical trial
  4000 patients

	Women	Men
Success	200	1800
Failure	1800	200

  - choose one patient unif. at random: each patient chosen w/prob 1/4000
  - A = {the patient is a Woman}
    B = {drug fails}
  - Is event A be independent from event B ?
  - Pr(A)=0.5, Pr(B)=0.5, Pr(AB)=9/20

Independence: examples



- Consider the experiment of tossing a coin twice
- Examples: is event A independent from event B?
  - A = {HT, HH}={Coin1=H}, B = {HT}
  - A = {HT}, B = {TH}
- Disjoint ≠ Independence
- If A is independent from B, B is independent from C, is A independent from C?

Not necessarily, say A=C

Conditional probability

- If A and B are events with Pr(A) > 0, ***conditional probability of B given A*** is
$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$
- Example: medical trial

	Women	Men
Success	200	1800
Failure	1800	200

Choose one patient at random
A = {Patient is a Woman}
B = {Drug fails}
Pr(B|A) = 18/20
Pr(A|B) = 18/20

- If A is independent from B, Pr(A|B)= P(A)

Conditional Independence

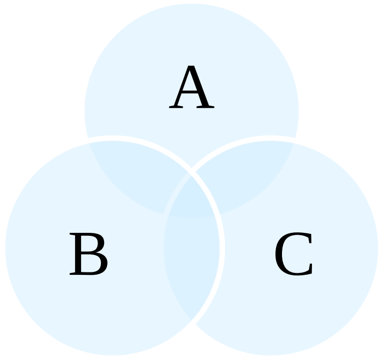
- Event A and B are ***conditionally independent given C*** if

  $$\Pr(AB|C) = \Pr(A|C)\Pr(B|C)$$

- Events $\{A_i\}$ are conditionally mutually independent given C if

  $$\Pr(\cap_i A_i|C) = \Pi_i \Pr(A_i|C)$$

Conditional Independence (cont'd)



- Example: three events A, B, C
  - $Pr(A) = Pr(B) = Pr(C) = 1/5$
    $Pr(AC) = Pr(BC) = 1/25$, $Pr(AB) = 1/10$
    $Pr(ABC) = 1/125$
  - Are A, B independent? $1/5*1/5 \neq 1/10$
  - Are A, B conditionally independent given C?

    $Pr(A|C)= (1/25)/(1/5)=1/5$,
    $Pr(B|C)= (1/25)/(1/5)=1/5$

    $Pr(AB|C)=(1/125)/(1/5)= 1/25= Pr(A|C)Pr(B|C)$
- A and B are independent
  $\neq$ A and B are conditionally independent

Random Variable

- ***Experiment***: e.g.: toss a coin twice
  - sample space S and probability $\Pr(\cdot)$
- A ***random variable*** X assigns a number to every outcome
  - X = #heads
  - "function from sample space to numbers"
  - shorthand: RV for "random variable"
- ***Distribution*** of X assigns probability $\Pr(X = x)$ to every $x \in \Re$
  - ***probability mass function*** (pmf) $f_X(x) = \Pr(X = x)$
- ***Support*** of X is the set of all $x \in \Re$ for which $f_X(x) > 0$

Random Variable: Example

- Experiment: three rolls of a die.
  Let X be the sum of #dots on the three rolls.
- What are the possible values for X?
- Pr(X = 3) = 1/6*1/6*1/6=1/216,
- Pr(X = 5) = ?

Expectation

- Expectation of random variable X

$$E[X] = \sum_x x \Pr(X = x)$$

  - weighted average of numbers in the support
- Nice properties:
  - $E[c] = c$ for any constant c.
  - Additive: $E[X + Y] = E[X] + E[Y]$
  - Linear: $E[\alpha X] = \alpha E[X]$ for any $\alpha \in \Re$
  - Monotone: if $X \leq Y$ with prob. 1, then $E[X] \leq E[Y]$

Conditional expectation

- ***Conditional expectation*** of RV X given event A:

$$E[X|A] = \sum_{x \in \mathrm{suport}} x \Pr(X = x|A)$$

  - same formula as $E[X]$, but with conditional probabilities
- = expectation of X in a "conditional" probability space
  - same sample space as before
  - all probabilities conditioned on A
- same nice properties as before

Variance

- ***Variance*** of RV X:
$$Var(X) = E\big((X - E[X])^2\big) = E(X^2) - (E[X])^2$$
  - characterizes how much X spreads away from its expectation
- Nice properties:
  - $Var(X) \geq 0$
  - $Var(X + c) = Var(X)$ for any constant c
  - $Var(\alpha X) = \alpha^2 Var(X)$ for any $\alpha \in \Re$
- ***standard deviation*** $\sigma(X) = \sqrt{Var(X)}$
- NB: variance can be infinite!
  - $X = 2^i$ with probability 2^{-i}, for each $i = 1,2,3, \ldots$.

Uniform distribution

- choose "uniformly at random" (u.a.r.)
  - sample space: K items
  - same probability $\frac{1}{K}$ for each item.
- (discrete) uniform distribution
  - random variable X can take K possible values
  - all values have the same probability $\frac{1}{K}$

Bernoulli & Binomial

- ***Bernoulli*** distribution
  - success with probability p, failure otherwise
  - ***Bernoulli*** RV X (a.k.a. 0-1 RV):
    $$\Pr(X = 1) = p \text{ and } \Pr(X = 0) = 1 - p$$
  - E[X] = p, Var(X) = $E[X^2]-E[X]^2 = p-p^2$
- ***Binomial distribution***
  - $X =$ #successes in n draws of a Bernoulli distribution
  - X_i~Bernoulli(p), $i = 1 \ldots n$
    $X = \sum_{i=1}^{n} X_i$, X~Bin(p, n)
  - E[X] = np, Var(X) = np(1-p)

Independent RVs

- Two random variables X and Y on the same experiment
  - outcomes of two coin tosses
- Joint distribution: $f_{X,Y}(x, y) = \Pr(X = x, Y = y)$
- X and Y are ***independent*** if for all $x, y \in \Re$

$$f_{X,Y}(x, y) = \Pr(X = x) \Pr(Y = y)$$

  - equiv.: if events $\{X = x\}$ and $\{Y = y\}$ are independent
- Basic properties:

$$E[XY] = E[X]\, E[Y]$$

$$\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$$

- RVs $X, Y, Z, \ldots$ ***mutually independent*** if

$$\Pr(X = x, Y = y, Z = z, \ldots) = \Pr(X = x) \Pr(Y = y) \Pr(Z = z) \ \ldots$$

- Shorthand: ***IID*** for "independent and identically distributed"

Outline

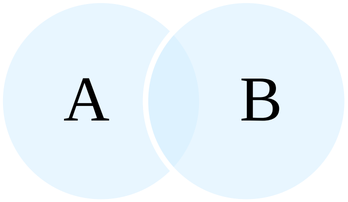
- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

Infinitely many outcomes

- experiments can have infinitely many outcomes
  - all finite sequences of coin tosses
  - *countably* many outcomes => same treatment as before
- experiments can have ***"continuously"*** many outcomes
  - throw a dart randomly into a unit interval
Outcomes: all numbers in [0,1]
  - infinite sequence of coin tosses
Outcomes: infinite binary sequences
- Sample space S: set of all possible outcomes
  - Events: subsets of S
- Probabilities assigned to events, not to individual outcomes!

Definition of Probability

- ***Probability of an event*** : a number assigned to event Pr(A)
  - Axiom 1: 0<= Pr(A) <= 1
  - Axiom 2: Pr(S) = 1, Pr(∅)= 0
  - Axiom 3: For any two events A and B,
    $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB)$



- Corollaries
  - $\Pr(\bar{A}) = 1 - \Pr(A)$
  - For every sequence of disjoint events $\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$

Probability space

- ***Probability space*** consists of three things:
  - sample space S
  - set of events $\mathcal{F}$ (where each event is s subset of S)
  - probability $\Pr(A)$ for each event $A \in \mathcal{F}$
- $\mathscr{F}$ is the set of events that "we care about"
  - OK to care about some, but not all events ($\mathscr{F}$ does not have to include all events)
  - $\mathscr{F}$ must satisfy some formal properties ("σ-algebra") to make probability well-defined

Random variable X

- ***Experiment***: infinite sequence of coin tosses
  - sample space: infinite binary sequences $(b_1, b_2, \ldots)$
- A ***random variable*** X assigns a number to every outcome
  - $X = 0.b_2 b_4 b_6 \ \ldots \in [0,1]$
  - "function from sample space to numbers"
- ***Distribution*** of X: assigns probability to every interval:

$$\Pr(a \leq X \leq b)$$

  - ***cumulative distribution function*** (cdf)

$$F_X(x) = \Pr(X \leq x)$$

Continuous vs discrete

- ***"Continuous"*** random variable X:
  - each possible value happens with zero probability
  - "throw a dart randomly into a unit interval"
- ***"Discrete"*** random variable Y:
  - each possible value happens with positive probability
  - #heads in two coin tosses
  - NB: may happen even if #outcomes is infinite, e.g.:

$$\Pr(Y = i) = 2^{-i}, \qquad i = 1,2,3,\ldots$$

- RVs can be neither "continuous" nor "discrete"! E.g., max(X,Y)

Probability density function (pdf)

- ***Pdf*** for random variable X is a function $f_X(x)$ such that

$$\Pr(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

  - not guaranteed to exist (but exists in many useful cases)
- ***Support*** of X = {all x such that $f_X(x) > 0$}
  - How to define "support" if pdf does not exist? E.g.:
    - Y is discrete random variable, and $Z = X$ with probability ½ , and $Z = Y$ otherwise.
    - Then support(Z) = support(X) ∪ support(Y)

Expectation

- If pdf f_X exists, then expectation is

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

- General definition (for any random variable)
  - Lebesgue integral of X with respect to measure $\Pr(\cdot)$
  - no need to know what it is, for this course
- Same nice properties as in the discrete case

Uniform distribution

- Informally:
  - “ Throw a random dart into an interval $[a, b]$ ”
  - “ each number has the same probability ”
- Formally:
  - sample space: all numbers in $[a, b]$
  - probability density function: $f_X(x) = 1/(b - a)$
  - equivalently:

$$\Pr(a' \leq X \leq b') = (b' - a')/(b - a)$$

  for every interval $[a', b'] \subset [a, b]$

Independent RVs

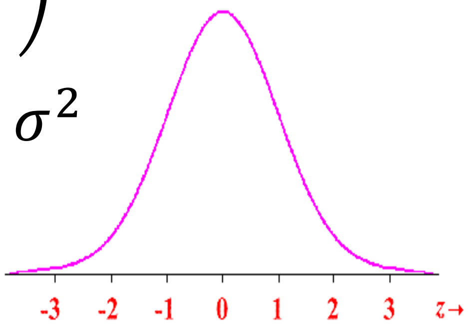
- Two random variables X and Y on the same experiment
  - " two throws of a dart into a unit interval "
- ***Joint distribution*** of X and Y
  assigns probability $\Pr(X \in I, Y \in J)$, for any two intervals I, J
- X and Y are independent if for all intervals I, J
  $$\Pr(X \leq x, Y \leq y) = \Pr(X \leq x)\Pr(Y \leq y)$$
  - equivalently: if events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent
- Random variables $X, Y, Z, \ldots$ ***mutually independent*** if
  $$\Pr(X \leq x, Y \leq y, Z \leq z, \ldots) = \Pr(X \leq x)\Pr(Y \leq y)\Pr(Z \leq z)\ \ldots$$

Normal (Gaussian) Distribution

- Random variable $X \sim N(\mu, \sigma^2)$ defined by pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

  - two parameters: expectation μ and variance σ^2
  - "standard normal distribution": $N(0,1)$



- Nice properties:
  - If $X_1 \sim N(\mu_1, \sigma^2_1)$ and $X_2 \sim N(\mu_2, \sigma^2_2)$ are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma^2_1 + \sigma^2_2)$
  - Central Limit Theorem (informally):
    If $Y_1, \ldots, Y_n$ are IID RVs with finite variance,
    their average converges to a normal distribution as $n \to \infty$

Outline

- Basics: “discrete” probability
- Basics: “continuous” probability
- Concentration inequalities

Concentration inequalities

- **Setup:** $X_1, \ldots, X_n$ independent random variables.
  (not necessarily identically distributed)
  $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ is the average, and $\mu = \mathbb{E}[\bar{X}]$
- Strong Law of Large Numbers:
  $$\Pr\left(\bar{X} \xrightarrow{n} \mu\right) = 1$$
- Want: $\bar{X}$ is ***concentrated*** around μ when n is large,
  i.e. that $|\bar{X} - \mu|$ is small with high probability.
  - $\Pr(|\bar{X} - \mu| \leq \text{"small"}) \geq 1 - \text{"small"}$
  - such statements are called “concentration inequalities”

Hoeffding Inequality (HI)

- High-prob. event: $\mathcal{E}_{\alpha,T} = \left\{ |\bar{X} - \mu| \le \sqrt{\frac{\alpha \log T}{n}} \right\}, \alpha \ge 0$
- **<u>HI:</u>** Assume $X_i \in [0,1]$ for all i. Then

$$\Pr(\mathcal{E}_{\alpha,T}) \ge 1 - 2T^{-2\alpha}.$$

  - $\alpha = 2$ suffices for most applications in this course.
    T controls probability; can be the time horizon in MAB
  - this is a convenient re-formulation of HI for our purposes
    more "flexible" and "generic" formulation exists
- "Chernoff Bounds": special case when $X_i \in \{0,1\}$
- Relevant notation: $r = \sqrt{\frac{\alpha \log T}{n}}$ "confidence radius

  $[\mu - r,\ \mu + r]$ "confidence interval"

Hoeffding Inequality (extensions)

- **Recall**: $\mathcal{E}_{\alpha,T} = \left\{ |\bar{X} - \mu| \le \sqrt{\frac{\alpha \log T}{n}} \right\}, \alpha \ge 0$
- **"HI for intervals":** Assume $X_i \in [a_i, b_i]$ for all i. Then

  $$\Pr(\mathcal{E}_{\alpha\beta,T}) \ge 1 - 2T^{-2\alpha}, \text{ where } \beta = \frac{1}{n}\sum_{i=1}^{n}(b_i - a_i)^2.$$
- **"HI for small variance":**
  Assume $X_i \in [0,1]$ and $Var(X_i) \le v$ for all i. Then

  $$\Pr\left(\mathcal{E}_{\alpha v, T}\right) \ge 1 - 2T^{-\alpha/4}.$$

  as long as n is large enough: $\frac{n}{\log n} \ge \frac{\alpha}{9v}$.
- **"HI for Gaussians":**
  Assume X_i is Gaussian with variance $\le v$.Then

  $$\Pr\left(\mathcal{E}_{\alpha v, T}\right) \ge 1 - 2T^{-\alpha/2}.$$

Concentration for non-independent RVs

- **Setup:** $X_1, \ldots, X_n$ random variables in [0,1] (***not necessarily independent*** or identically distributed) $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ is the average
- **Assume:** there is a number $\mu_i \in [0,1]$ such that
$$E(X_i | X_1 \in J_1, \ldots, X_{i-1} \in J_{i-1}) = \mu_i$$
for any intervals $J_1, \ldots, J_{i-1} \subset \Re$.

for each $i = 1, \ldots, n$

- **Let** $\mathcal{E}_\alpha = \left\{ |\bar{X} - \mu| \leq \sqrt{\frac{\alpha \log T}{n}} \right\}, \alpha \geq 0$

$\mu = (\mu_1 + \cdots + \mu_n)/n$

- Then $\Pr(\mathcal{E}_{\alpha,T}) \geq 1 - 2T^{-\alpha/2}$

Follows from Azuma-Hoeffding inequality

- If $\mu_1 = \cdots = \mu_n = 0$, sequence $(X_1, \ldots, X_n)$ is called a ***martingale***

Union bound

Setup: finite or countable set of events $A_1, A_2, \ldots$

Then $\Pr[\bigcup_i A_i] \leq \sum_i \Pr[A_i]$

Resources

- A survey on concentration inequalities
  by Fan Chung and Linyuan Lu (2010)
- Another survey on concentration inequalities
  by Colin McDiarmid (1998).
- Wikipedia
  - Hoeffding inequality
  - Azuma-Hoeffding inequality